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THE CONSTRUCTION OF JACOBI AND PERIODIC JACOBI MATRICES WITH PRESCRIBED SPECTRA

Warren E. Ferguson, Jr.

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ABSTRACT

The spectral properties of Jacobi and periodic Jacobi matrices are analyzed and algorithms for the construction of Jacobi and periodic Jacobi matrices with prescribed spectra are presented. Numerical evidence demonstrates that these algorithms are of practical utility. These algorithms have been used in studies of the periodic Toda lattice, and might also be used in studies of inverse eigenvalue problems for Sturm-Louiville equations and Hill's equation.

AMS (MOS) Subject Classification: 15A18, 65F15, 65F30, 65L15

Key Words: Inverse eigenvalue problem, tridiagonal matrix, Jacobi matrix, periodic tridiagonal matrix, periodic Jacobi matrix, Floquet theory, resolvent identities.

Work Unit Number 7 - Numerical Analysis

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Significance and Explanation

In this report we present algorithms which solve two inverse eigenvalue problems that arise in matrix theory. Computational evidence is presented that demonstrates that these algorithms are of practical utility.

The first inverse eigenvalue problem considers what additional information uniquely determines the entries of a Jacobi matrix if we know its eigenvalues. Recall that a Jacobi matrix is a real, symmetric tridiagonal matrix whose next to diagonal entries are positive.

The second inverse eigenvalue problem considers what additional information uniquely determines the entries of a periodic Jacobi matrix if we know its eigenvalues. A periodic Jacobi matrix is obtained by replacing the entries in the upper right and lower left corners of a Jacobi matrix by the same positive number.

Inverse eigenvalue problems of this nature arise in mathematical physics. For example, the construction of a linear array of masses interconnected by springs with prescirbed normal modes of vibration leads to such inverse eigenvalue problems. In addition, the construction of a ladder network of inductors and capacitors with prescribed transmission characteristics also leads to such inverse eigenvalue problems.

Finally, FORTRAN subroutines which implement these algorithms are presented in an appendix.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE CONSTRUCTION OF JACOBI AND PERIODIC JACOBI MATRICES

WITH PRESCRIBED SPECTRA

Warren E. Ferguson, Jr.

1. Introduction. A periodic Jacobi matrix is any real, symmetric matrix of the form

$$L = \begin{bmatrix} a_1 & b_1 & & b_N \\ b_1 & \ddots & \ddots \quad 0 & \\ & 0 \quad \ddots & \ddots & b_{N-1} \\ b_N & & b_{N-1} & a_N \end{bmatrix} \qquad \text{where} \quad b_i > 0 \quad \forall\, i .$$

This paper shows how one can construct a periodic Jacobi matrix with prescribed spectra. For example, there is a family of periodic Jacobi matrices with $\lambda_1, \cdots, \lambda_N$ as eigenvalues if and only if the numbers $\lambda_1, \cdots, \lambda_N$ are real and can be ordered so that

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \cdots \quad .$$

Similar problems have been studied by other authors [2, 12].

The results presented in this paper are based upon an analysis of the spectral properties of periodic Jacobi matrices. The main tool in this analysis is the knowledge of the spectral properties of Jacobi matrices. Recall that a Jacobi matrix is any real, symmetric tridiagonal matrix whose next to diagonal entries are positive. Our cannonical Jacobi matrix will be the matrix obtained by deleting from L the last row and column, that is

$$J = \begin{bmatrix} a_1 & b_1 & & 0 \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-2} \\ 0 & & b_{N-2} & a_{N-1} \end{bmatrix} \qquad \text{where} \quad b_i > 0 \quad \forall\, i.$$

An algorithm which constructs a Jacobi matrix with prescribed spectra is presented in Theorem 2. This algorithm is derived from the fact that any real, symmetric matrix has real eigenvalues and a corresponding full set of real, orthonormal eigenvectors. We hasten to point out that essentially the same algorithm was presented by de Boor and Golub [3]. Similar problems have been studied by other authors [8, 9].

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The spectral properties of periodic Jacobi matrices are considered in Section 3. The results presented in this section are derived from a matrix analog of Floquet theory [11]. In Section 4 we use these results to characterize the family of periodic Jacobi matrices with prescribed spectra.

We present the results of several numerical experiments in Section 5. These results demonstrate that the algorithms presented in Theorems 2 and 6 are of practical utility. Indeed, these algorithms have been used in performing numerical experiments on the periodic Toda lattice [4]. In Section 6 we conclude the paper with several comments.

2. Spectral Properties of Jacobi Matrices. In this section we will consider the spectral properties of the Jacobi matrix

$$J = \begin{bmatrix} a_1 & b_1 & & 0 \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-2} \\ 0 & & b_{N-2} & a_{N-1} \end{bmatrix} \qquad \text{where } b_i > 0 \quad \forall\, i .$$

Observe that J is a real, symmetric matrix. Consequently J has real eigenvalues $\mu_1,\cdots,\mu_{N-1}$ and a corresponding set $Y_1,\cdots,Y_{N-1}$ of real, orthonormal eigenvectors [13,14]. If Y denotes the matrix whose jth column is Y_j then Y is an orthogonal matrix and

$$JY = YD \qquad \text{where} \qquad D = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_{N-1} \end{bmatrix}. \tag{1}$$

Many important relationships between the eigenvalues and eigenvectors of J can be derived from the representation

$$(\mu I - J)^{-1} = Y(\mu I-D)^{-1}\, Y^T \tag{2}$$

of the resolvent of J. For example, by comparing the entries in row 1, column N-1 of (2) we arrive at the identity

$$b_1 \cdots b_{N-2} = \sum_k{}_1^{N-1} \frac{\omega(\mu)}{\mu-\mu_k}\, Y_{1,k}\, Y_{N-1,k}\,. \tag{3.a}$$

Here

$$\omega(\mu) = \det\,(\mu I-J) \tag{3.b}$$

is the characteristic polynomial of J and $Y_{i,j}$ denotes the entry of Y in row i, column j. Another important identity, used by Stieltjes in his treatment of inverse eigenvalue problems, can be derived from (2) by comparing the entries in row 1, column 1 (or row N-1, column N-1.)

In the work that follows we will demonstrate that the entries of J can be recovered from the entries on the diagonal of D and in the first row of Y. Before we describe this process let us introduce the following:

Definition 1: (a) The Jacobi matrix J is characterized by the data $\{\mu, y\}$ if and only if

(1) $\mu_1, \cdots, \mu_{N-1}$ are the eigenvalues of J, and

(2) $y_1, \cdots, y_{N-1}$ are the first components of a set $Y_1, \cdots, Y_{N-1}$ of real, orthonormal eigenvectors of J corresponding to $\mu_1, \cdots, \mu_{N-1}$.

(b) The data $\{\mu, y\}$ is compatible if and only if

(1) $\mu_1, \cdots, \mu_{N-1}$ are real, distinct numbers, and

(2) $y_1, \cdots, y_{N-1}$ are real, nonzero numbers whose squares sum to one.

We feel justified in using the words "characterize" and "compatible" in this manner because the following theorem is true.

Theorem 2: Data characterizing a Jacobi matrix is compatible. Furthermore each set of compatible data $\{\mu, y\}$ characterizes a unique Jacobi matrix J. The entries (a, b) of this Jacobi matrix are computed by the algorithm:

1. $Y_{0,j} = 0 \quad \forall j$
2. $Y_{1,j} = y_j \quad \forall j$
3. For $i = 1, \cdots, N-2$
4. $\quad a_i = \sum_{k}^{N-1} {}_1 \mu_k Y_{i,k}^2$
5. $\quad b_i^2 = \sum_{k}^{N-1} {}_1 [(\mu_k - a_i) Y_{i,k} - b_{i-1} Y_{i-1,k}]^2$
6. $\quad Y_{i+1,j} = \frac{1}{b_i}[(\mu_j - a_i) Y_{i,j} - b_{i-1} Y_{i-1,j}] \quad \forall j$
7. Next i
8. $a_{N-1} = \sum_{k}^{N-1} {}_1 \mu_k Y_{N-1,k}^2$.

Proof: The proof of this theorem will be presented as a sequence of three lemmas. ■

Lemma 2.1: Data characterizing a Jacobi matrix is compatible.

Proof: Let the Jacobi matrix J be characterized by the data $\{\mu, y\}$. The μ's are necessarily real because they are the eigenvalues of a real, symmetric matrix. By definition the y's are real, and their squares sum to one because they may be considered to be the entries in the first row of the orthogonal matrix Y in (1). Consider the limiting form of the identity (3) as μ tends to μ_j. If μ_j were a repeated eigenvalue then $\omega'(\mu_j) = 0$ and so we would be forced to conclude that $b_1 \cdots b_{N-2} = 0$, which is impossible because each $b_i > 0$. Therefore the μ's are distinct and, as μ tends to μ_j, we infer that

$$b_1 \cdots b_{N-2} = \omega'(\mu_j)\, Y_{1,j} Y_{N-1,j} \qquad \forall j . \tag{4}$$

Consequently the y's are nonzero because $y_j = Y_{1,j}$. ∎

Lemma 2.2: Given compatible data $\{\mu, y\}$ the algorithm of Theorem 2 computes the entries (a,b) of a Jacobi matrix J characterized by the data $\{\mu, y\}$.

Proof: First, we infer that this algorithm computes the entries (a,b) of some Jacobi matrix J only if the value of b_i computed in step 5 is never zero. From the compatibility of the data we infer that $b_1 > 0$. If $b_1, \cdots, b_{\ell-1} > 0$ but $b_\ell = 0$ for some $\ell < N-1$ then step 6 implies

$$\begin{bmatrix} a_1 & b_1 & & 0 \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{\ell-1} \\ 0 & & b_{\ell-1} & a_\ell \end{bmatrix} \begin{bmatrix} Y_{1,j} \\ \vdots \\ Y_{\ell,j} \end{bmatrix} = \mu_j \begin{bmatrix} Y_{1,j} \\ \vdots \\ Y_{\ell,j} \end{bmatrix} \qquad \forall j .$$

But this is impossible, for no matrix of order $\ell < N-1$ has $N-1$ distinct eigenvalues.

Second, we will demonstrate that the numbers $Y_{i,j}$ computed by this algorithm form the entries of an orthogonal matrix Y, that is the rows of Y satisfy the orthonormality relations

$$\sum_{k=1}^{N-1} Y_{i,k} Y_{j,k} = \delta_{i,j} \qquad \text{for} \quad j = 1, \cdots, i \tag{5}$$

and $i = 1, \cdots, N-1$. From the compatibility of the data $\{\mu, y\}$ we infer that (5) is true for $i = 1$. If (5) is true for $i = 1, \cdots, \ell$ then the following argument demonstrates that it is also true for $i = \ell+1$. Clearly steps 5 and 6 imply that (5) is true for $j = \ell+1$. For $j \le \ell$ step 6 implies that

$$\sum_{k=1}^{N-1} Y_{\ell+1,k} Y_{j,k} = \frac{1}{b_\ell} \Big[\sum_{k=1}^{N-1} \mu_k Y_{\ell,k} Y_{j,k} - a_\ell \delta_{\ell,j} - b_{\ell-1} \delta_{\ell-1,j} \Big] .$$

The right side of this equality is zero for $j = \ell$ because step 4 was executed, and it is zero for $j < \ell$ because step 6 implies that

$$\sum_{k=1}^{N-1} \mu_k Y_{\ell,k} Y_{j,k} = \sum_{k=1}^{N-1} Y_{\ell,k}[b_{j-1}Y_{j-1,k} + a_j Y_{j,k} + b_j Y_{j+1,k}] = b_j \delta_{\ell,j+1} .$$

Third, we will demonstrate that the data $\{\mu,y\}$ characterizes J. It will be sufficient to prove that the matrices J,Y constructed by this algorithm satisfy (1). Step 6 implies that $JY = YD$ if we can show that the numbers

$$Y_{N,j} \equiv (\mu_j - a_{N-1})Y_{N-1,j} - b_{N-2}Y_{N-2,j} \qquad \forall\ j$$

are zero. The techniques presented in the previous paragraph can be used to demonstrate that

$$\sum_{k=1}^{N-1} Y_{N,k} Y_{j,k} = 0 \qquad \text{for } j = 1,\cdots,N-1 .$$

Since the rows of Y form a real, orthonormal basis we infer that

$$Y_{N,j} = 0 \qquad \forall\ j$$

■

Lemma 2.3: Each set of compatible data characterizes at most one Jacobi matrix.

Proof: Let $\hat{J}$ be any Jacobi matrix characterized by the compatible data $\{\mu,y\}$. Then $y_1,\cdots,y_{N-1}$ are the first components of a set $\hat{Y}_1,\cdots,\hat{Y}_{N-1}$ of real, orthonormal eigenvectors of $\hat{J}$ corresponding to the eigenvalues $\mu_1,\cdots,\mu_{N-1}$. If $\hat{Y}$ denotes the matrix whose $j^{\underline{th}}$ column is $\hat{Y}_j$ then $\hat{Y}$ is an orthogonal matrix and

$$\hat{J}\,\hat{Y} = \hat{Y}\,D \qquad \text{where} \qquad \hat{D} = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_{N-1} \end{bmatrix} .$$

We will now prove that the entries $(\hat{a},\hat{b})$ of $\hat{J}$ are identical to the entries (a,b) of the Jacobi matrix J computed by the algorithm presented in Theorem 2.

The entries $\hat{Y}_{i,j}$ of $\hat{Y}$ satisfy the orthonormality relations

$$\sum_{k=1}^{N-1} \hat{Y}_{i,k}\hat{Y}_{j,k} = \delta_{i,j} \qquad \forall\ i,j$$

because $\hat{Y}\hat{Y}^T = I$. The entries $\hat{Y}_{i,j}$ of Y also satisfy the recurrence relation

$$\hat{b}_{i-1}\hat{Y}_{i-1,j} + \hat{a}_i\hat{Y}_{i,j} + \hat{b}_i\hat{Y}_{i+1,j} = \mu_j\hat{Y}_{i,j} \qquad \forall\ i,j$$

where

$$\hat{Y}_{0,j} = 0 \quad \text{and} \quad \hat{Y}_{N,j} = 0 \qquad \forall\ j$$

because $\hat{J}\,\hat{Y} = \hat{Y}\,D$. When the recurrence relation is multiplied by $\hat{Y}_{i,j}$ and the result is summed over j we find, using the orthonormality relations, that

$$\hat{a}_i = \sum_{k=1}^{N-1} \mu_k \hat{Y}^2_{i,k} .$$

The recurrence relation also implies that

$$\hat{Y}_{i+1,j} = \frac{1}{\hat{b}_i}[(\mu_j - \hat{a}_i)\hat{Y}_{i,j} - \hat{b}_{i-1}\hat{Y}_{i-1,j}] \qquad \forall j$$

and, when this identity is squared and the result is summed over j, the orthonormality relations imply that

$$\hat{b}_i^2 = \sum_{k=1}^{N-1}[(\mu_k - \hat{a}_i)\hat{Y}_{i,k} - \hat{b}_{i-1}\hat{Y}_{i-1,k}]^2 .$$

Starting with the fact that

$$\hat{Y}_{1,j} = y_j \qquad \forall j$$

it is easily shown by induction, following the sequence of computations presented in the algorithm, that the entries $(\hat{a},\hat{b})$ of $\hat{J}$ are identical to the entries (a,b) of J.

■

3. Spectral Properties of Periodic Jacobi Matrices. In this section we will consider the spectral properties of the periodic Jacobi matrix

$$L = \begin{bmatrix} a_1 & b_1 & & & b_N \\ b_1 & \ddots & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{N-1} \\ b_N & & & b_{N-1} & a_N \end{bmatrix} \qquad \text{where} \quad b_i > 0 \quad \forall i .$$

Throughout this section we will use J to represent the Jacobi matrix obtained by deleting from L the last row and column.

Observe that L is a real, symmetric matrix. Consequently L has real eigenvalues and corresponding set of real, orthonormal eigenvectors [13, 14]. Let z be an eigenvector of L corresponding to the eigenvalue λ. Then the components z_i of z form a nontrivial solution of the recurrence relation $(b_0 \equiv b_N)$

$$b_{i-1}z_{i-1} + a_i z_i + b_i z_{i+1} = \lambda z_i \qquad \forall i$$

which satisfies the boundary conditions

$$z_N = z_0 \qquad \text{and} \qquad z_{N+1} = z_1 .$$

By analogy with Floquet theory, which analyzes the analogous problem for ordinary differential equations [11], let us consider the nontrivial solutions of the recurrence relation which satisfy the boundary conditions

$$z_N = \rho z_0 \qquad \text{and} \qquad z_{N+1} = \rho z_1 .$$

Here the parameter ρ is called the Floquet multiplier of z. This problem has only the trivial solution when $\rho = 0$, while for $\rho \neq 0$ a nontrivial solution exists if and only if λ is an eigenvalue of the matrix

$$L_\rho = \begin{bmatrix} a_1 & b_1 & & & \frac{1}{\rho} b_N \\ b_1 & \ddots & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{N-1} \\ \rho b_N & & & b_{N-1} & a_N \end{bmatrix}$$

With these facts in mind let us introduce the following:

Definition 3: Let J be characterized by the data $\{\mu, y\}$ and have $\omega(\mu)$ as its characteristic polynomial. Then the Floquet multipliers $\rho_1, \cdots, \rho_{N-1}$ of L corresponding to $\mu_1, \cdots, \mu_{N-1}$ are the numbers defined by the relation

$$b_1 \cdots b_N = -\rho_j \omega'(\mu_j)\ b_N^2 y_j^2 \qquad \forall\ j . \tag{6}$$

Theorem 4: The characteristic polynomial of L_ρ admits the representation

$$\det(\lambda I - L_\rho) = b_1 \cdots b_N \{\Delta(\lambda) - (\rho + \frac{1}{\rho})\} \tag{7}$$

where $\Delta(\lambda)$, called the discriminant of L, is independent of ρ. The Floquet multipliers $\rho_1, \cdots, \rho_{N-1}$ of L corresponding to the eigenvalues $\mu_1, \cdots, \mu_{N-1}$ of J satisfy the relation

$$\Delta(\mu_j) = \rho_j + \frac{1}{\rho_j} \qquad \forall\ j . \tag{8}$$

The eigenvalues $\lambda_1, \cdots, \lambda_N$ of L are real and can be ordered so that

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \cdots .$$

Proof: Using elementary properties of determinants it is not hard to demonstrate that

$$\frac{d}{d\rho} \det(\lambda I - L_\rho) = -b_1 \cdots b_N \ (1 - \frac{1}{\rho^2}) .$$

When both sides are integrated with respect to ρ we find that

$$\det\ (\lambda I - L_\rho) = b_1 \cdots b_N \{\Delta(\lambda) - (\rho + \frac{1}{\rho})\} .$$

Of course the constant of integration $b_1 \cdots b_N\ \Delta(\lambda)$ is necessarily independent of ρ.

Let J be characterized by the data $\{\mu, y\}$. Then $y_1, \cdots, y_{N-1}$ are the first components of a set $Y_1, \cdots, Y_{N-1}$ of real, orthonormal eigenvectors of J corresponding to its eigenvalues $\mu_1, \cdots, \mu_{N-1}$. Let $Y_{i,j}$ denote the ith component of Y_j. From the definition (6) of the Floquet multipliers and the identity (4) we infer that

$$\rho_j = -\frac{b_{N-1} Y_{N-1,j}}{b_N Y_{1,j}} \qquad \forall\ j$$

and so

$$L_{\rho_j} \begin{bmatrix} Y_j \\ \\ 0 \end{bmatrix} = \mu_j \begin{bmatrix} Y_j \\ \\ 0 \end{bmatrix} \qquad \forall j .$$

Consequently μ_j is an eigenvalue of L_{ρ_j} for each j and we infer from (7) that (8) is true.

From the definition (6) of the Floquet multipliers we deduce that

$$\omega'(\mu_j)\rho_j < 0 \qquad \forall j .$$

When the eigenvalues of J are ordered so that

$$\mu_1 > \cdots > \mu_{N-1}$$

then we infer from (8) that

$$(-1)^j \Delta(\mu_j) \geq 2 \qquad \forall j$$

because the magnitude of $\rho + \frac{1}{\rho}$ is never less than two. Consequently the eigenvalues $\lambda_1,\cdots,\lambda_N$ of L, which are the roots of $\Delta(\lambda) = 2$, are real and can be ordered so that

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \cdots$$

because the coefficient $(b_1\cdots b_N)^{-1}$ of λ^N in $\Delta(\lambda)$ is positive. ■

A typical discriminant for a periodic Jocobi matrix L of order $N = 6$ is illustrated in Figure 1. In this figure we depict the relationship between the eigenvalues $\lambda_1,\cdots,\lambda_N$ of L and the Floquet multipliers $\rho_1,\cdots,\rho_{N-1}$ of L corresponding to the eigenvalues $\mu_1,\cdots,\mu_{N-1}$ of J .

Let us introduce the following:

Definition 5: (a) The periodic Jacobi matrix L is characterized by the data $\{A,B,\mu,\rho\}$ if and only if

(1) $A = a_1 + \cdots + a_N$,

(2) $B = b_1 \cdots b_N$,

(3) $\mu_1,\cdots,\mu_{N-1}$ are the eigenvalues of J , and

(4) $\rho_1,\cdots,\rho_{N-1}$ are the Floquet multipliers of L corresponding to $\mu_1,\cdots,\mu_{N-1}$.

(b) The data $\{A,B,\mu,\rho\}$ is compatible if and only if

(1) A is a real number,

(2) B is a real, positive number,

(3) $\mu_1,\cdots,\mu_{N-1}$ are real, distinct numbers, and

(4) $\rho_1,\cdots,\rho_{N-1}$ are real numbers which satisfy $\omega'(\mu_j)\rho_j < 0 \quad \forall j$ with $\omega(\mu) = (\mu-\mu_1)\cdots(\mu-\mu_{N-1})$.

We feel justified in using the words "characterize" and "compatible" in this manner because the following theorem is true.

Theorem 6: Data characterizing a periodic Jacobi matrix is compatible. Furthermore, each set of compatible data $\{A,B,\mu,\rho\}$ characterizes a unique periodic Jacobi matrix L. The entries (a,b) of this periodic Jacobi matrix are computed by the algorithm:

1. $$b_N^2 = -\sum_{k=1}^{N-1} \frac{B}{\rho_k \omega'(\mu_k)}$$

2. $$y_j = \frac{1}{b_N}\sqrt{-\frac{B}{\rho_j \omega'(\mu_j)}} \quad \forall j$$

3. Recover J from the data $\{\mu,y\}$

4. $$b_{N-1} = \frac{B}{b_1 \cdots b_{N-2} b_N}$$

5. $$a_N = A - (a_1 + \cdots + a_{N-1})$$

with $\omega(\mu) = (\mu-\mu_1)\cdots(\mu-\mu_{N-1})$.

Proof: The proof of this theorem will be presented as a sequence of three lemmas.

■

Lemma 6.1: Data characterizing a periodic Jacobi matrix is compatible.

Proof: Let the periodic Jacobi matrix L be characterized by the data $\{A,B,\mu,\rho\}$. Clearly A is a real number because it is a sum of real numbers, while B is a real, positive number because it is a product of real, positive numbers. The μ's are real, distinct numbers because they are the eigenvalues of the Jacobi matrix J. while the definition (6) of the ρ's makes it obvious that they are real, nonzero numbers which satisfy $\omega'(\mu_j)\rho_j < 0$ for all j because $\omega(\mu)$ is also the characteristic polynomial of J.

■

Lemma 6.2: Given compatible data $\{A,B,\mu,\rho\}$ the algorithm of Theorem 6 computes the entries (a,b) of a periodic Jacobi matrix L characterized by the data $\{A,B,\mu,\rho\}$.

Proof: The data $\{\mu,y\}$ used in step 3 is compatible, therefore it is clear that this algorithm computes the entries (a,b) of some periodic Jacobi matrix L. Let L be characterized by the data $\{\hat{A},\hat{B},\hat{\mu},\hat{\rho}\}$. From steps 4 and 5 it is clear that $\hat{A} = A$ and $\hat{B} = B$. In view of Theorem 2 we know that J is characterized by the data $\{\mu,y\}$. Therefore $\hat{\mu}_j = \mu_j$ for all j and from the definition of the Floquet multipliers we know that

$$B = -\hat{\rho}_j \omega'(\mu_j) \; b_N^2 \, y_j^2 \qquad \forall j .$$

Step 2 therefore implies that $\hat{\rho}_j = \rho_j$ for all j.

■

Lemma 6.3: Each set of compatible data characterizes at most one periodic Jacobi matrix.

Proof: Let $\hat{L}$ be any periodic Jacobi matrix characterized by the compatible data $\{A,B,\mu,\rho\}$. Let the Jacobi matrix $\hat{J}$, obtained by deleting from $\hat{L}$ the last row and column, be characterized by the data $\{\mu,\hat{y}\}$. Without loss of generality we may assume that each $\hat{y}_j$ is positive, for if $\hat{y}_j$ is the first component of an eigenvector $\hat{Y}_j$ of $\hat{J}$ then $-\hat{y}_j$ is the first component of the eigenvector $-\hat{Y}_j$ of $\hat{J}$. We will now prove that the entries $(\hat{a},\hat{b})$ of $\hat{L}$ are identical to the entries (a,b) of the periodic Jacobi matrix L constructed by the algorithm of Theorem 6.

By definition the Floquet multipliers $\hat{\rho}_1,\cdots,\hat{\rho}_{N-1}$ of $\hat{L}$ corresponding to $\mu_1,\cdots,\mu_{N-1}$ satisfy the relationship

$$B = -\rho_j \omega'(\mu_j)\ \hat{b}_N^2\, \hat{y}_j^2 \qquad \forall\ j\ .$$

The sum of the squares of the $\hat{y}$'s equals one because the data $\{\mu,\hat{y}\}$ is compatible, therefore

$$\hat{b}_N^2 = -\sum_{k}^{N-1} \frac{B}{\rho_k \omega'(\mu_k)} \qquad \text{and} \qquad \hat{y}_j = \frac{1}{b_N}\sqrt{-\frac{B}{\rho_j \omega'(\mu_j)}} \quad \forall\ j\ .$$

In view of steps 1 and 2 we infer that $\hat{b}_N = b_N$ and $\hat{y}_j = y_j$ for all j. Since both $\hat{J}$ and J are characterized by the same data then Theorem 2 implies that $\hat{J} = J$. Finally, steps 4 and 5 imply that $\hat{b}_{N-1} = b_{N-1}$ and $\hat{a}_N = a_N$.

■

4. Periodic Jacobi Matrices with Prescribed Spectra. With these basic facts established let us now consider how we can characterize the family of periodic Jacobi matrices whose eigenvalues are $\lambda_1,\cdots,\lambda_N$.

Let L be a periodic Jacobi matrix characterized by the data $\{A,B,\mu,\rho\}$. Then $\lambda_1,\cdots,\lambda_N$ are the eigenvalues of L *if and only if* the discriminant $\Delta(\lambda)$ of L admits the representation

$$\Delta(\lambda) = 2 + \frac{1}{B}(\lambda-\lambda_1)\cdots(\lambda-\lambda_N)\ .$$

Therefore the problem of characterizing the family of periodic Jacobi matrices with prescribed spectra is intimately related to the problem of characterizing the family of periodic Jacobi matrices with prescribed discriminant. Let us introduce the following:

Definition 7: For each polynomial $p(\lambda)$ let $F(p)$ denote the family of periodic Jacobi matrices whose discriminant is $p(\lambda)$.

The problem of characterizing which periodic Jacobi matrices belong to $F(p)$ is answered in the following:

Theorem 8: Let $p(\lambda)$ be a polynomial of degree N. The data $\{A,B,\mu,\rho\}$ characterizes a member of $F(p)$ if and only if:

(1) the data $\{A,B,\mu,\rho\}$ is compatible,

(2) $p(\lambda) = \frac{1}{B}[\lambda^N - A\lambda^{N-1} + \text{lower powers of } \lambda]$, and

(3) $p(\mu_j) = \rho_j + \frac{1}{\rho_j} \quad \forall j$.

Furthermore, $F(p)$ is nonempty if and only if

(4) the coefficient of λ^N in $p(\lambda)$ is positive, and

(5) $p(\lambda)$ has local extrema at $N-1$ real, distinct points $\nu_1 > \cdots > \nu_{N-1}$ with $(-1)^j p(\nu_j) \geq 2 \quad \forall j$.

Proof: The proof of this theorem will be presented as a sequence of two lemmas.

■

Lemma 8.1: The data $\{A,B,\mu,\rho\}$ characterizes a member of $F(p)$ if and only if conditions (1,2,3) of Theorem 8 are satisfied.

Proof: If the data $\{A,B,\mu,\rho\}$ characterizes a member of $F(p)$ then Theorems 4 and 6 demonstrate that conditions (1,2,3) of Theorem 8 are satisfied.

Let us now suppose that conditions (1,2,3) of Theorem 8 are satisfied. Let $\Delta(\lambda)$ be the discriminant of the periodic Jacobi matrix characterized by the data $\{A,B,\mu,\rho\}$. Now

$$q(\lambda) \equiv \Delta(\lambda) - p(\lambda)$$

is a polynomial of degree $N-2$ because the coefficients of λ^{N-1}, λ^N in $\Delta(\lambda)$, $p(\lambda)$ agree. Theorem 4 also implies that $q(\mu_j) = 0 \quad \forall j$ and so

$$q(\lambda) \equiv 0$$

because the only polynomial of degree $N-2$ which is zero at $N-1$ distinct points is the trivial polynomial. Consequently the data $\{A,B,\mu,\rho\}$ characterizes a member of $F(p)$.

■

Lemma 8.2: $F(p)$ is nonempty if and only if conditions (4,5) of Theorem 8 are satisfied.

Proof: If $F(p)$ is nonempty then Lemma 8.1 and the mean-value theorem can be used to demonstrate that conditions (4,5) of Theorem 8 are satisfied.

Let us now suppose that conditions (4,5) of Theorem 8 are satisfied. Let A,B be determined so that

$$p(\lambda) = \frac{1}{B}[\lambda^N - A\lambda^{N-1} + \text{lower powers of } \lambda]$$

and $\rho_1,\cdots,\rho_{N-1}$ be solutions of

$$p(\nu_j) = \rho_j + \frac{1}{\rho_j} \quad \forall\, j .$$

Then the data $\{A,B,\nu,\rho\}$ is compatible and from Lemma 8.1 we infer that the data $\{A,B,\nu,\rho\}$ characterizes a member of $F(p)$.

∎

Using Theorem 8 it is not hard to prove the following:

Corollary 9: The periodic Jacobi matrix L has $\lambda_1,\cdots,\lambda_N$ as its eigenvalues if and only if

$$L \in \bigcup_{B>0} F(\Delta_B)$$

where $\Delta_B(\lambda) \equiv 2 + \frac{1}{B}(\lambda-\lambda_1)\cdots(\lambda-\lambda_N)$. Furthermore, there is a periodic Jacobi matrix with $\lambda_1,\cdots,\lambda_N$ as its eigenvalues if and only if the numbers $\lambda_1,\cdots,\lambda_N$ can be ordered so that

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \cdots .$$

5. Numerical Experiments. Let us now present the results of several numerical experiments. These experiments were carried out on a UNIVAC 1110 in single precision floating point arithmetic (27 bit mantissa) using FORTRAN versions of the algorithms presented in Theorems 2 and 6 .

In the first experiment we test the algorithm presented in Theorem 2. The results of this experiment are presented in Table 1. Observe that this algorithm has difficulty in recovering the Jacobi matrix described in Example 3.

Experiment 1:

1. Select a Jacobi matrix J of order $N-1$.
2. Compute the data $\{\mu,y\}$ characterizing J [13, 14, 15]:
   (a) use bisection to compute the μ's , and
   (b) use inverse iteration to compute the y's .
3. Use the algorithm presented in Theorem 2 to reconstruct the Jacobi matrix $\hat{J}$ characterized by the data $\{\mu,y\}$.
4. Output the error $\|J - \hat{J}\|$ where

$$\|A\| = \max_{i,j} |a_{i,j}| .$$

In the second experiment we test the algorithm presented in Theorem 6. The results of this experiment are presented in Table 2. Observe that the Jacobi matrices used in the examples of Experiment 1 are obtained by deleting the last row and column from the periodic Jacobi matrix used in the corresponding examples of Experiment 2.

Experiment 2:

1. Select a periodic Jacobi matrix L of order N .
2. Compute the data $\{A,B,\mu,\rho\}$ characterizing L :
   (a) use the obvious sum to compute A ,
   (b) use the obvious product to compute B ,
   (c) compute the data $\{\mu,y\}$ characterizing J as described in Step 2 of Experiment 1, and
   (d) compute the ρ's using Equation (6).
3. Use the algorithm presented in Theorem 6 to reconstruct the periodic Jacobi matrix $\hat{L}$ characterized by the data $\{A,B,\mu,\rho\}$.
4. Output the error $\|L-\hat{L}\|$ where

$$\|A\| = \max_{i,j} |a_{ij}| \ .$$

In both of these experiments we have not worked with matrices of order $N > 30$. The reason why we have not worked with matrices of order $N > 30$ may be explained as follows. In Example 2 of Experiment 2 some of the components of y in the data $\{\mu,y\}$ become smaller as N increases. For example, the smallest component of y changes from 2×10^{-9} for $N = 15$ to 2×10^{-20} for $N = 30$. Since the Floquet multipliers ρ depend on the squares of the data y we will run into underflow problems when $N > 30$. The immediate remedy for this underflow problem is the use of logarithms in the computation of the Floquet multipliers. However, underflow also occurs in the computation of y when $N > 55$, consequently the use of logarithms is not a panacea.

6. Comments. Let $\tilde{\omega}(\mu)$ be the characteristic polynomial of the Jacobi matrix $\tilde{J}$ obtained from J by deleting the first row and column. By comparing the entries of (2) in row 1, column 1 we find that

$$\tilde{\omega}(\mu) = \sum_{k=1}^{N-1} \frac{\omega(\mu)}{\mu-\mu_k} Y_{1,k}^2 \ .$$

This identity was used by Stieltjes in his study of inverse eigenvalue problems. As μ tends to μ_j we deduce that

$$\tilde{\omega}(\mu_j) = \omega'(\mu_j) Y_{1,j}^2 \qquad \forall j \ .$$

From this identity we infer that the eigenvalues of $\tilde{J}$ strictly interlace those of J . Furthermore, from the eigenvalues of J and $\tilde{J}$ we can recover the data $\{\mu,y\}$ characterizing J and hence J itself.

It is interesting to note that the algorithm presented in Theorem 2 is used in some versions of the implicit shift QR algorithm [13]. These versions of the QR algorithm make use of the fact that if $B = QAQ^H$, where B is an unreduced upper Hessenberg matrix and Q is a unitary matrix, then the entries of B and Q are uniquely determined from the entries of A and the entries in the first row of Q . In our application $A = D$, $B = J$ and $Q = Y$.

We can also recover the Jacobi matrix J from the eigenvalues and the last components of the corresponding real, orthonormal eigenvectors of J. To understand why let us consider the permutation matrix

$$S = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} .$$

We find that $S^2 = I$, therefore from Equation (1) we deduce that

$$(SJS)(SY) = (SY)D .$$

Consequently the algorithm presented in Theorem 2 states that the entries of

$$SJS = \begin{bmatrix} a_{N-1} & b_{N-2} & & 0 \\ b_{N-2} & \ddots & \ddots & \\ & \ddots & \ddots & b_1 \\ 0 & & b_1 & a_1 \end{bmatrix}$$

can be recovered from the entries of D and the entries in the first row of SY, that is the last row of Y.

It also appears that Theorem 2 can be extended to some class of band matrices. For example, let the real, symmetric matrix

$$K = \begin{bmatrix} a_1 & b_1 & c_1 & & 0 \\ b_1 & \ddots & \ddots & \ddots & \\ c_1 & \ddots & \ddots & \ddots & c_{N-3} \\ & \ddots & \ddots & \ddots & b_{N-2} \\ 0 & & c_{N-3} & b_{N-2} & a_{N-1} \end{bmatrix} \quad \text{where } c_i > 0 \ \forall i .$$

have $\mu_1, \cdots, \mu_{N-1}$ as eigenvalues and $Y_1, \cdots, Y_{N-1}$ as the corresponding set of real, orthonormal eigenvectors. If Y denotes the matrix whose $j^{\underline{th}}$ column is Y_j then Y is an orthogonal matrix and

$$KY = YD \quad \text{where} \quad D = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_{N-1} \end{bmatrix} .$$

Following the argument presented in Lemma 2.3 we arrive at an algorithm which recovers K from the entries in D and in the first two rows of Y.

The paper by Golub and Welsch [7] outlines how one can modify the usual QR algorithm and compute directly the data $\{\mu, y\}$ characterizing a Jacobi matrix J. Furthermore, their paper also presents a matrix version of the celebrated Gelfand-Levitan solution to the inverse eigenvalue problem for a class of Sturm-Liouville problems.

The paper by Kammerer [10] describes an algorithm that can be used to construct a discriminant whose "shape" is prescribed. By the "shape" of a discriminant we are referring to the value of the discriminant at each of its $N-1$ real, distinct local extrema. For applications of Kammerer's algorithm to the periodic Toda lattice we refer the reader to the forthcoming paper [4].

Useful information concerning properties of periodic Jacobi matrices is contained in [1]. We would also like to state that the analysis presented in Section 3 can be extended in the same generality to "anti-periodic" Jacobi matrices of the form

$$L_{-1} = \begin{bmatrix} a_1 & b_1 & & -b_N \\ b_1 & \ddots & \ddots & 0 \\ & \ddots & \ddots & b_{N-1} \\ -b_N & 0 & b_{N-1} & a_N \end{bmatrix} \qquad \text{where } b_i > 0 \quad \forall\, i .$$

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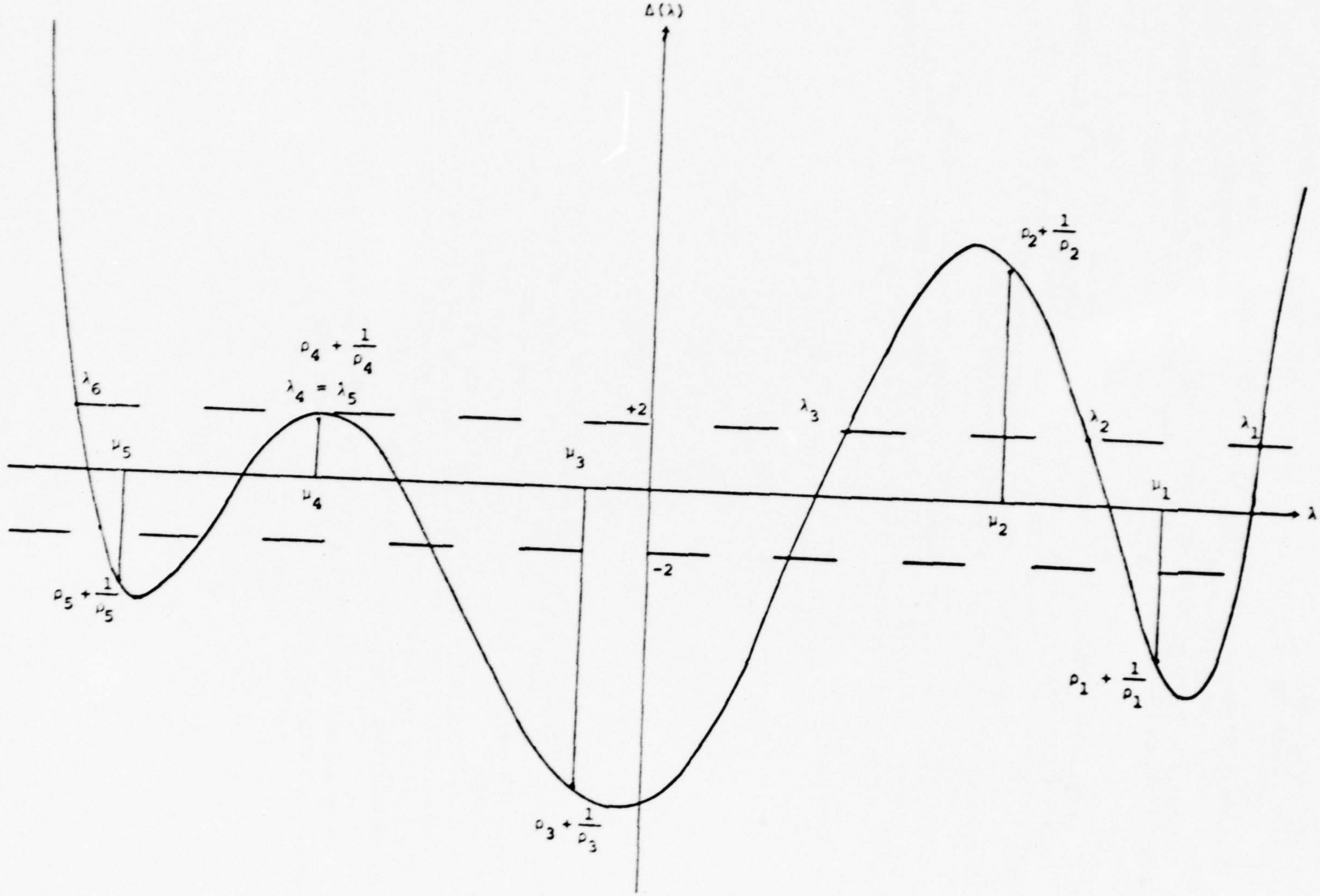

Figure 1: Plot of a Typical Discriminant, N=6

Example 1:

$A(I) = -2 \qquad I = 1,\cdots,N-1$

$B(I) = 1 \qquad I = 1,\cdots,N-2$

N	Error
5	4×10^{-8}
10	2×10^{-7}
15	5×10^{-7}
20	2×10^{-7}
25	2×10^{-7}
30	6×10^{-7}

Example 2:

$A(I) = (N+1-I)/N-2 \qquad I = 1,\cdots,N-1$

$B(I) = 1-(N-I)/N \qquad I = 1,\cdots,N-2$

N	Error
5	4×10^{-8}
10	1×10^{-7}
15	4×10^{-7}
20	3×10^{-7}
25	3×10^{-7}
30	9×10^{-7}

Table 1: Results of Experiment 1.

Example 3:

$$A(I) = I/N - 2 \qquad I = 1,\cdots,N-1$$

$$B(I) = 1 - I/N \qquad I = 1,\cdots,N-2$$

N	Error
5	1×10^{-7}
10	3×10^{-7}
15	2×10^{-4}
20	2×10^{0}
25	2×10^{0}
30	1×10^{0}

Table 1: Results of Experiment 1.

Example 1:

$$A(I) = -2 \qquad I = 1,\cdots,N-1$$

$$B(I) = 1 \qquad I = 1,\cdots,N-2$$

$$A(N) = 0$$

$$B(N-1) = B(N) = 1$$

N	Error
5	9×10^{-8}
10	5×10^{-7}
15	1×10^{-6}
20	2×10^{-6}
25	3×10^{-6}
30	5×10^{-6}

Example 2:

$$A(I) = (N+1-I)/N-2 \qquad I = 1,\cdots,N-1$$

$$B(I) = 1-(N-I)/N \qquad I = 1,\cdots,N-2$$

$$A(N) = 0$$

$$B(N-1) = B(N) = 1$$

N	Error
5	1×10^{-7}
10	2×10^{-7}
15	4×10^{-7}
20	3×10^{-7}
25	6×10^{-7}
30	4×10^{-7}

Table 2 - Results of Experiment 2

Example 3:

$$A(I) = I/N-2 \qquad I = 1,\cdots,N-1$$
$$B(I) = 1 - I/N \qquad I = 1,\cdots,N-2$$
$$A(N) = 0$$
$$B(N-1) = B(N) = 1$$

N	Error
5	4×10^{-8}
10	1×10^{-7}
15	1×10^{-3}
20	2×10^{0}
25	4×10^{0}
30	5×10^{0}

Table 2. Results of Experiment 2.

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8. Appendix

In this section we present listings of several FORTRAN subroutines that the author used while performing various computational experiments.

No warranties, expressed or implied, are made by the author that this program is free of error. It should not be relied on as the sole basis to solve a problem whose incorrect solution could result in injury to person or property. If the program is employed in such a manner, it is at the user's own risk and the author disclaims all liability for such misuse.

```
        ===== DTABL =====

        SUBROUTINE DTABL(N,X,V,P)
        REAL X(N),V(N)
        COMMON /DIVTB/ KDERV,NPTS,XNODE(51),DIVDF(51)
C
C        CONSTRUCT THE DIVIDED DIFFERENCE TABLE *NPTS,XNODE,DIVDF*
C        BASED ON THE *N* NODES *X* AND THE CORRESPONDING FUNCTIO
C        VALUES *V*.  THE DIVIDED DIFFERENCE TABLE IS CENTERED
C        AT THE POINT *P*
C
C        INITIALIZE
C
        NPTS = N
        DO 10 I=1,NPTS
         XNODE(I) = X(I)
         DIVDF(I) = V(I)
     10 CONTINUE
C
C        BUBBLE SORT THE NODES
C
        DO 30 I=2,NPTS
         DO 20 K=2,I
          J = I-K+1
          IF (ABS(P-XNODE(J)).GE.ABS(P-XNODE(J+1))) GO TO 30
           EXCH = XNODE(J)
           XNODE(J) = XNODE(J+1)
           XNODE(J+1) = EXCH
           EXCH = DIVDF(J)
           DIVDF(J) = DIVDF(J+1)
           DIVDF(J+1) = EXCH
     20  CONTINUE
     30 CONTINUE
C
C        SET UP THE DIVIDED DIFFERENCE TABLE
C
        DO 50 J=2,NPTS
         DO 40 K=J,NPTS
          I = K-J+1
          DIVDF(I) = (DIVDF(I+1)-DIVDF(I))/(XNODE(K)-XNODE(I))
     40  CONTINUE
     50 CONTINUE
C
        RETURN
C
        END
```

===== EIGEN =====

```
      SUBROUTINE EIGEN(N,A,B,MU,Y,D,W)
      REAL A(N),B(N),MU(N),Y(N),D(N),W(N),LO,HI,MID
      DATA NITS/10/
C
C      COMPUTE THE EIGNEVALUES *MU* AND THE FIRST COMPONENTS
C      *Y* OF ORTHONORMAL EIGENVECTORS OF THE JACOBI MATRIX
C
C           A(1)   B(1)                    0
C           B(1)     .      .
C                    .      .      .
C                           .      .   B(N-1)
C            0                  B(N-1)  A(N)
C
C      D,W ... WORKING STORAGE VECTORS OF LENGTH  N .
C
      NM1 = N-1
C
C      COMPUTE THE MACHINE EPSILON
C
      EPS = 1.
   10 EPS = EPS/2.
      TEST = 1.+EPS
      IF (TEST.GT.1.) GO TO 10
      EPS = 2.*EPS
C
C      COMPUTE GERSCHGORIN BOUNDS FOR THE EIGENVALUES
C
      GMIN = A(1) - B(1)
      GMAX = A(1) + B(1)
      DO 20 I=2,NM1
       GMIN = AMIN1(GMIN,A(I) - (B(I-1)+B(I)))
       GMAX = AMAX1(GMAX,A(I) + (B(I-1)+B(I)))
   20 CONTINUE
      GMIN = AMIN1(GMIN,A(N)-B(NM1))
      GMAX = AMAX1(GMAX,A(N)+B(NM1))
      SIZE = AMAX1(ABS(GMIN),ABS(GMAX))
C
C      COMPUTE *MU* AND *Y*
C
      DO 120 I=1,N
C
C      COMPUTE *MU(I)* BY BISECTION
C
C      *NUM* , THE NUMBER OF NEGATIVE DIAGONAL ENTRIES IN
C      THE LU FACTORIZATION OF *MID - JACOBI* , COUNTS THE
C      NUMBER OF EIGENVALUES OF THE JACOBI MATRIX GREATER
C      THAN *MID*
C
       LO = GMIN
       HI = GMAX
   30  DIF = HI-LO
       TEST = SIZE + DIF/2.
       MID = LO + DIF/2.
       IF (TEST.LE.SIZE) GO TO 50
       TEST = EPS*SIZE
       NUM = 0
       D(1) = MID - A(1)
```

===== EIGEN =====

```
            IF (ABS(D(1)).LT.TEST) D(1) = SIGN(TEST,D(1))
            IF (D(1).LT.0.) NUM = NUM+1
            DO 40 J=2,N
             D(J) = (MID - A(J)) - B(J-1)**2/D(J-1)
             IF (ABS(D(J)).LT.TEST) D(J) = SIGN(TEST,D(J))
             IF (D(J).LT.0.) NUM = NUM+1
   40       CONTINUE
            IF (NUM.GE.I) LO = MID
            IF (NUM.LT.I) HI = MID
            GO TO 30
C
C           COMPUTE *Y(I)* BY INVERSE ITERATION
C
C           THE DIAGONAL ENTRIES OF THE LU FACTORIZATION OF
C           *MID - JACOBI* HAVE BEEN COMPUTED ABOVE
C
   50       W(1) = 1.
            DO 60 J=2,N
             W(J) = 0.
   60       CONTINUE
            DO 110 IT=1,NITS
             DO 70 J=1,NM1
              W(J+1) = W(J+1) + B(J)*W(J)/D(J)
   70        CONTINUE
             W(N) = W(N)/D(N)
             DO 80 JR=1,NM1
              J = N-JR
              W(J) = (W(J) + B(J)*W(J+1))/D(J)
   80        CONTINUE
             SUM = 0.
             DO 90 J=1,N
              SUM = SUM + W(J)**2
   90        CONTINUE
             SUM = SQRT(SUM)
             DO 100 J=1,N
              W(J) = W(J)/SUM
  100        CONTINUE
  110       CONTINUE
C
            MU(I) = MID
            Y(I) = W(1)
C
  120 CONTINUE
C
      RETURN
C
      END
```

===== FALSI =====

```
      FUNCTION FALSI(A,B,TOL,F,VALUE,IFLAG)
C
C       IN THE INTERVAL BETWEEN *A* AND *B* COMPUTE TO AN ACCURACY
C       *TOL* THE LOCATION *FALSI* WHERE *F* ASSUMES *VALUE*.
C       IF UPON RETURN
C
C       IFLAG = 1 ... THEN *FALSI* WAS FOUND TO AN ACCURACY OF *TOL*
C               2 ... THEN *TOL* WAS NEGATIVE
C               3 ... THEN F(X)-VALUE HAS THE SAME SIGN AT X=A,B
C
C       IF *TOL* IS ZERO THEN *FALSI* IS FOUND TO MACHINE PRECISION
C
      IFLAG = 1
C
C       CHECK LEFT ENDPOINT FOR A ZERO
C
      FALSI = AMIN1(A,B)
      F1 = F(FALSI) - VALUE
      IF (F1.EQ.0.) RETURN
      X1 = FALSI
      TEST = F1
C
C       CHECK RIGHT ENDPOINT FOR A ZERO
C
      FALSI = AMAX1(A,B)
      F2 = F(FALSI) - VALUE
      IF (F2.EQ.0.) RETURN
      X2 = FALSI
C
C       CHECK FOR REASONABLE *TOL*
C
      IF (TOL.GE.0.) GO TO 10
       IFLAG = 2
       RETURN
C
C       CHECK FOR SIGN CHANGE
C
   10 IF (F1*SIGN(1.,F2).LT.0.) GO TO 20
       IFLAG = 3
       RETURN
C
C       LINEAR INTERPOLATION USED TO COMPUTE APPROXIMATE LOCATION
C       OF THE ZERO
C
   20 SAVE = TEST
      RATIO = F1/(F1-F2)
      FALSI = X1 + RATIO*(X2-X1)
C
C       CHECK FOR TERMINATION
C
      IF ( (X2-X1).LE.TOL*AMAX1(ABS(X1),ABS(X2)) ) RETURN
C
C       CHECK IF GUESS FOR ZERO IS ACCEPTABLE
C
      IF ( (FALSI.GT.X1).AND.(FALSI.LT.X2) ) GO TO 30
C
C       IF THE MIDPOINT IS ALSO UNACCEPTABLE THEN *TOL* IS TOO SMALL
```

===== FALSI =====

```
C           AND *FALSI* IS THE BEST THAT CAN BE DONE
C
             FALSI = X1 + .5*(X2-X1)
             IF ( (FALSI.LE.X1).OR.(FALSI.GE.X2) ) RETURN
C
C           UPDATE INFORMATION
C
        30 TEST = F(FALSI) - VALUE
           IF (TEST*SIGN(1.,F1)) 40,50,60
C
        40   X2 = FALSI
             F2 = TEST
             IF (TEST*SIGN(1.,SAVE).GT.0.) F1 = .5*F1
             GO TO 20
C
        50   RETURN
C
        60   X1 = FALSI
             F1 = TEST
             IF (TEST*SIGN(1.,SAVE).GT.0.) F2 = .5*F2
             GO TO 20
C
           END
```

```
          ***** JACOB *****

      SUBROUTINE JACOB(N,A,B,MU,Y,W1,W2)
      REAL A(N),B(N),MU(N),Y(N),W1(N),W2(N)
C
C       CONSTRUCT THE ENTRIES  (A,B)  OF A JACOBI MATRIX
C
C             A(1)    B(1)                      0
C             B(1)      .       .
C                       .       .       .
C                               .       .     B(N-1)
C             0                       B(N-1)  A(N)
C
C       WHERE  Y(J)  IS THE FIRST COMPONENT OF A NORMALIZED
C       EIGENVECTOR OF THE JACOBI MATRIX CORRESPONDING TO THE
C       EIGENVALUE  MU(J) .
C
C       WE ASSUME THAT THE  MU'S  ARE REAL,DISTINCT NUMBERS AND
C       THE  Y'S  ARE REAL,NONZERO NUMBERS WHOSE SQUARES SUM TO
C       ONE.
C
C       W1,W2 ... WORKING STORAGE VECTORS OF LENGTH  N .
C
C       IF THE USER DOES NOT NEED  Y  TO BE SAVED THEN CALL THIS
C       PROGRAM WITH THE  W1  ARGUMENT EQUAL TO  Y .
C
      NM1 = N-1
C
C       INITIALIZE
C
      DO 10 I=1,N
       W1(I) = Y(I)
       W2(I) = 0.
   10 CONTINUE
C
      BIM1 = 0.
      DO 50 I=1,NM1
C
C       COMPUTE *A(I)*
C
       A(I) = 0.
       DO 20 K=1,N
        A(I) = A(I) + MU(K)*W1(K)**2
   20  CONTINUE
C
C       COMPUTE *B(I)*
C
       B(I) = 0.
       DO 30 K=1,N
        T = (MU(K)-A(I))*W1(K) - BIM1*W2(K)
        B(I) = B(I) + T**2
   30  CONTINUE
       B(I) = SQRT(B(I))
C
C       COMPUTE THE NEXT *Y*
C
       DO 40 K=1,N
        T = (MU(K)-A(I))*W1(K) - BIM1*W2(K)
        W2(K) = W1(K)
```

```
        W1(K) = T/B(I)
   40   CONTINUE
C
        BIM1 = B(I)
   50 CONTINUE
C
C       COMPUTE *A(N)*
C
      A(N) = 0.
      DO 60 K=1,N
        A(N) = A(N) + MU(K)*W1(K)**2
   60 CONTINUE
C
      RETURN
C
      END
```

```
      =====  KAMMER  =====

      SUBROUTINE KAMMER(NP1,X,V,W1,IFLAG)
      REAL X(NP1),V(NP1),W1(NP1)
      COMMON /DIVTB/ KDERV,NPTS,XNODE(51),DIVDF(51)
      DATA PI/3.14159265358/
      EXTERNAL POLYV
C
C     GIVEN OSCILLATING DATA *V* COMPUTE THE NODES *X* OF
C     THE UNIQUE INTERPOLATING POLYNOMIAL *P(X)* FOR WHICH
C
C     1)  0.=X(NP1) .LT. ... .LT. X(1)=1.
C     2)  P(X(I)) = V(I)              FOR I=1,...,NP1
C     3)  DP(X(I))/DX = 0.            FOR I=2,...,NP1-1 .
C
C     IF UPON RETURN
C
C     IFLAG = 1 ... THEN EVERYTHING WORKED
C             2 ... THEN THE DATA *V* DOES NOT OSCILLATE .
C
C     W1 ... WORKING STORAGE VECTOR OF LENGTH  NP1 .
C
      N = NP1-1
C
C     CHECK THAT THE DATA *V* OSCILLATES
C
      IFLAG = 2
      DP = V(2) - V(1)
      IF (DP.EQ.0.) RETURN
      DO 10 I=2,N
       DM = DP
       DP = V(I+1) - V(I)
       IF (DP*SIGN(1.,DM).GE.0.) RETURN
   10 CONTINUE
C
C      INITIALIZE
C
      IFLAG = 1
      X(1) = 1.
      DO 20 I=2,N
       X(I) = .5*(1.+COS((I-1)*PI/N))
   20 CONTINUE
      X(NP1) = 0.
      CALL DTABL(NP1,X,V,0.)
C
C      COMPUTE THE MID-V POINTS
C
   30 SAVE = DIVDF(1)
      KDERV = 0
      DO 40 I=1,N
       XMID = X(I) + .5*(X(I+1)-X(I))
       CALL DTABL(NP1,X,V,XMID)
       VMID = V(I) + .5*(V(I+1)-V(I))
       W1(I) = FALST(X(I),X(I+1),0.,POLYV,VMID,IERR)
   40 CONTINUE
C
C      COMPUTE THE ZEROS OF THE DERIVATIVE
C
      KDERV = 1
```

===== KAMMER =====

```
      DO 50 I=2,N
       CALL DTABL(NP1,X,V,X(I))
       W1(I-1) = FALSI(W1(I-1),W1(I),0.,POLYV,0.,IERR)
   50 CONTINUE
C
C      UPDATE THE NODES
C
      DO 60 I=2,N
       X(I) = W1(I-1)
   60 CONTINUE
C
C      TEST FOR COMPLETION
C
      CALL DTABL(NP1,X,V,0.)
      IF (ABS(SAVE).LE.ABS(DIVDF(1))) RETURN
      GO TO 30
C
      END
```

===== PEIGEN =====

```
      SUBROUTINE PEIGEN(N,A,B,MU,RHO,W1,W2)
      REAL A(N),B(N),MU(N),RHO(N),W1(N),W2(N)
C
C      COMPUTE THE EIGENVALUES *MU* OF THE LEADING PRINCIPLE
C      SUBMATRIX OF THE PERIODIC JACOBI MATRIX
C
C           A(1)    B(1)                          B(N)
C           B(1)     .        .          0
C                    .        .          .
C                    0        .          .     B(N-1)
C           B(N)                     B(N-1)   A(N)
C
C      AND THEIR CORRESPONDING FLOQUET MULTIPLIERS *RHO*.  THE
C      SUM OF THE  A'S  IS STORED IN  MU(N)  AND THE PRODUCT
C      OF THE  B'S  IS STORED IN  RHO(N) .
C
C      W1,W2 ... WORKING STORAGE VECTORS OF LENGTH  N .
C
      NM1 = N-1
C
C      COMPUTE  MU(N)  AND  RHO(N)
C
      MU(N) = 0.
      RHO(N) = 1.
      DO 10 I=1,N
       MU(N) = MU(N) + A(I)
       RHO(N) = RHO(N)*B(I)
   10 CONTINUE
C
C      COMPUTE THE EIGENVALUES *MU* AND THE FIRST COMPONENTS
C      *RHO* OF ORTHONORMAL EIGENVECTORS OF THE LEADING
C      PRINCIPLE SUBMATRIX
C
      CALL EIGEN(NM1,A,B,MU,RHO,W1,W2)
C
C      REPLACE *RHO* BY THE FLOQUET MULTIPLIERS CORRESPONDING TO
C      THE EIGENVALUES *MU* .
C
      DO 30 I=1,NM1
       PROD = 1.
       DO 20 J=1,NM1
        IF (I.NE.J) PROD = PROD*(MU(I)-MU(J))
   20  CONTINUE
       RHO(I) = -RHO(N)/(PROD*(B(N)*RHO(I))**2)
   30 CONTINUE
C
      RETURN
C
      END
```

```
=====  PJACOB  =====

      SUBROUTINE PJACOB(N,A,B,MU,RHO,W1,W2)
      REAL A(N),B(N),MU(N),RHO(N),W1(N),W2(N)
C
C       CONSTRUCT THE ENTRIES  (A,B)  OF A PERIODIC JACOBI MATRIX
C
C           A(1)   B(1)                       B(N)
C           B(1)     .       .       0
C                    .       .       .
C                    0       .       .   B(N-1)
C           B(N)                 B(N-1)  A(N)
C
C       WHERE  RHO(J)  IS THE FLOQUET MULTIPLIER CORRESPONDING TO
C       THE EIGENVALUE  MU(J)  OF THE LEADING PRINCIPLE SUBMATRIX.
C       THE SUM OF THE  A'S  IS STORED IN  MU(N)  AND THE PRODUCT
C       OF THE  B'S  IS STORED IN  RHO(N) .
C
C       WE ASSUME THAT THE SUM OF THE  A'S  IS REAL, THE PRODUCT
C       OF THE  B'S  IS REAL AND POSITIVE, THE  MU'S ARE REAL AND
C       DISTINCT, AND THE  RHO'S  ARE REAL AND NONZERO NUMBERS
C       WHICH SATISFY
C                          '
C                     OMEGA ( MU(J) )*RHO(J)  .LT.  0.
C       FOR ALL  J  WHERE
C                   OMEGA(X) = (X-MU(1))*...*(X-MU(N-1))  .
C
C       W1,W2 ... WORKING STORAGE VECTORS OF LENGTH  N .
C
C       IF THE USER DOES NOT NEED  RHO  TO BE SAVED THEN CALL THIS
C       PROGRAM WITH THE  W1  ARGUMENT EQUAL TO  RHO  .
C
      NM1 = N-1
      NM2 = N-2
C
C       RECOVER *B(N)*
C
      DO 10 I=1,NM2
       W2(I) = MU(I+1)
   10 CONTINUE
      B(N) = 0.
      DO 30 K=1,NM1
       DERV = 1.
       DO 20 I=1,NM2
        DERV = DERV*(MU(K)-W2(I))
   20  CONTINUE
       B(N) = B(N) - RHO(N)/(RHO(K)*DERV)
       W2(K) = MU(K)
   30 CONTINUE
      B(N) = SQRT(B(N))
C
C       RECOVER THE FIRST COMPONENTS *W1* OF THE ORTHONORMAL
C       EIGENVECTORS OF THE LEADING PRINCIPLE SUBMATRIX.
C
      DO 40 I=1,NM2
       W2(I) = MU(I+1)
   40 CONTINUE
      DO 60 K=1,NM1
       DERV = 1.
```

===== PJACOB =====

```
          DO 50 I=1,NM2
           DERV = DERV*(MU(K)-W2(I))
   50     CONTINUE
          W1(K) = SQRT(-RHO(N)/(RHO(K)*DERV))/B(N)
          W2(K) = MU(K)
   60 CONTINUE
C
C         RECOVER THE LEADING PRINCIPLE SUBMATRIX
C
      CALL JACOB(NM1,A,B,MU,W1,W1,W2)
C
C         RECOVER *B(N-1)*
C
      B(NM1) = RHO(N)/B(N)
      DO 70 K=1,NM2
       B(NM1) = B(NM1)/B(K)
   70 CONTINUE
C
C         RECOVER *A(N)*
C
      A(N) = MU(N) - A(NM1)
      DO 80 K=1,NM2
       A(N) = A(N) - A(K)
   80 CONTINUE
C
      RETURN
C
      END
```

```
=====  POLYV  =====
      FUNCTION POLYV(X)
      COMMON /DIVTB/ KDERV,NPTS,XNODE(51),DIVDF(51)
C
C      COMPUTE THE VALUE *POLYV* OF THE INTERPOLATING POLYNOMIAL
C      *KDERV=0* OR ITS DERIVATIVE *KDERV=1* AT *X* BASED ON THE
C      DESCRIPTION GIVEN IN THE DIVIDED DIFFERENCE TABLE
C      *NPTS,XNODE,DIVDF*
C
C      HORNER'S ALGORITHM IS USED TO COMPUTE THE VALUE REQUIRED.
C
      POLY = 0.
      DPOLY = 0.
      DO 10 I=1,NPTS
       DPOLY = POLY + (X-XNODE(I))*DPOLY
       POLY = DIVDF(I) + (X-XNODE(I))*POLY
   10 CONTINUE
C
      POLYV = POLY
      IF (KDERV.EQ.1) POLYV = DPOLY
C
      RETURN
C
      END
```

===== RSCALE =====

```
      SUBROUTINE RSCALE(NP1,X,V)
      REAL X(NP1),V(NP1)
      COMMON /DIVTB/ KDERV,NPTS,XNODE(51),DIVDF(51)
C
C      SCALE THE DATA *X,V* SO THE DESCRIMINANT IT DESCRIBES
C      CHARACTERIZES A PERIODIC JACOBI MATRIX WHOSE SUM OF THE
C      A'S  IS ZERO AND WHOSE PRODUCT OF THE  B'S  IS  2**(-N) .
C
      N = NP1-1
      SUM = 0.
      DO 10 I=2,NP1
       SUM = SUM + X(I)
   10 CONTINUE
C
      CALL DTABL(NP1,X,V,X(NP1))
      SHIFT = (DIVDF(2)-SUM*DIVDF(1))/(N*DIVDF(1))
      EXPAN = EXP(-(N*ALOG(2.)-ALOG(DIVDF(1)))/N)
      DO 20 I=1,NP1
       X(I) = EXPAN*(X(I)+SHIFT)
   20 CONTINUE
C
      RETURN
      END
```

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The spectral properties of Jacobi and periodic Jacobi matrices are analyzed and algorithms for the construction of Jacobi and periodic Jacobi matrices with prescribed spectra are presented. Numerical evidence demonstrates that these algorithms are of practical utility. These algorithms have been used in studies of the periodic Toda lattice, and might also be used in studies of inverse eigenvalue problems for Sturm-Louiville equations and Hill's equation.

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